INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 3, 213–216 (1983)

GENERATING EXACT SOLUTIONS OF THE TWO-DIMENSIONAL BURGERS' EQUATIONS

CLIVE A. J. FLETCHER

Department of Mechanical Engineering, University of Sydney, Sydney, N.S.W. 2006 Australia

KEY WORDS Burgers' Equations Exact Solution Fluid Flow

Burgers' equation is well suited to modelling fluid flows as it incorporates directly the interaction between the non-linear convection processes and the diffusive viscous processes. In one dimension the Cole–Hopf procedure transforms Burgers' equation into the linear heat conduction equation. As a result many exact solutions of Burgers' equation are available in the literature. Thus Burgers' equation has often been used as a model equation for comparing the accuracy of different computational algorithms. This aspect of Burgers' equation is reviewed by Fletcher.[1]

The two-dimensional Burgers' equations

$$u_t + uu_x + vu_y - \frac{1}{R_e}(u_{xx} + u_{yy}) = 0 \tag{1}$$

$$v_t + uv_x + vv_y - \frac{1}{R_e}(v_{xx} + v_{yy}) = 0 \tag{2}$$

are the same as the incompressible Navier–Stokes equations with the pressure gradient terms removed. Clearly, solutions to equations (1) and (2) would not, necessarily, satisfy the continuity equation. However, the two-dimensional Burgers' equations constitute an appropriate model for developing computational algorithms, for solving the incompressible Navier–Stokes equations. Numerical solutions of equations (1) and (2) have been obtained[2,3] but the accuracy of the results were assessed indirectly via grid refinement as no exact solutions were available to the authors.

Here a procedure is developed for generating exact solutions of the two-dimensional Burgers' equations. It has been pointed out by Cole[4] and Ames[5] that the Cole–Hopf transformation can be interpreted as a multi-dimensional transformation. In two dimensions the Cole–Hopf transformation relates a function, ϕ, to u and v in the following way.

$$u = \frac{-2}{R_e}\frac{\phi_x}{\phi} \tag{3}$$

and

$$v = \frac{-2}{R_e}\frac{\phi_y}{\phi}. \tag{4}$$

Then equations (1) and (2) become

$$\phi_t - (\phi_{xx} + \phi_{yy}) = 0. \tag{5}$$

0271–2091/83/030213–04$01.00

Received 9 October 1981
Revised 28 May 1982

For simplicity attention will be focussed on the steady counterparts of equations (1), (2) and (5) i.e.

$$\phi_{xx} + \phi_{yy} = 0. \tag{6}$$

In one dimension the equation (which is closely related to Burgers' equation),

$$(u - 0{\cdot}5)u_x - \frac{1}{\mathrm{R_e}} u_{xx} = 0, \tag{7}$$

has the solution

$$u = 0{\cdot}5 - 0{\cdot}5 \tanh(0{\cdot}25\ \mathrm{R_e}\ x). \tag{8}$$

The solution (8) represents a shock wave centred at $x = 0$ and the shock wave can be made sharper (larger gradient, u_x) by increasing $\mathrm{R_e}$. The structure of $\tanh(kx)$ and the form of equation (3) suggest that a suitable contribution to ϕ might be

$$\Delta\phi = (\exp(kx) + \exp(-kx)) \cos(ky), \tag{9}$$

which satisfies equation (6).

To provide more control over the 'velocity' distributions, equation (9) is generalized to the following,

$$\phi = a_0 + a_1 x + a_2 y + a_3 xy + \{\exp(k(x - x_0)) + \exp(-k(x - x_0))\} \cos(ky). \tag{10}$$

It can be verified by substitution that equation (10) satisfies equation (6). Using equations (3) and (4) the following expressions for u and v are obtained,

$$u = \frac{-2}{\mathrm{R_e}} \frac{\{a_1 + a_3 y + k[\exp(k(x - x_0)) - \exp(-k(x - x_0))] \cos ky\}}{\{a_0 + a_1 x + a_2 y + a_3 xy + [\exp(k(x - x_0)) + \exp(-k(x - x_0))] \cos ky\}} \tag{11}$$

and

$$v = \frac{-2}{\mathrm{R_e}} \frac{\{a_2 + a_3 x - k[\exp(k(x - x_0)) + \exp(-k(x - x_0))] \sin ky\}}{\{a_0 + a_1 x + a_2 y + a_3 xy + [\exp(k(x - x_0)) + \exp(-k(x - x_0))] \cos ky\}}. \tag{12}$$

If $a_0 = a_1 = a_2 = a_3 = 0$ and $x - x_0 \gg 0$, equations (11) and (12) become

$$u = -2k/\mathrm{R_e}, \qquad v = (2k \tan ky)/\mathrm{R_e}. \tag{13}$$

If $a_0 = a_1 = a_2 = a_3 = 0$ and $x - x_0 \ll 0$, equations (11) and (12) become

$$u = 2k/\mathrm{R_e}, \qquad v = (2k \tan ky)/\mathrm{R_e}. \tag{14}$$

A typical evaluation of equations (11) and (12) for the domain $-1 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant y_{max}$ is

Table I. Exact solution for u of two-dimensional Burgers' equation

y/y_{max}	$x = -1{\cdot}0$	$-0{\cdot}6$	$-0{\cdot}2$	$0{\cdot}2$	$0{\cdot}6$	$1{\cdot}0$
1·0	0·9988	0·9748	0·7483	0·1407	−0·0862	−0·0992
0·8	0·9989	0·9761	0·7588	0·1529	−0·0841	−0·0992
0·6	0·9989	0·9770	0·7666	0·1623	−0·0825	−0·0991
0·4	0·9990	0·9777	0·7718	0·1690	−0·0813	−0·0991
0·2	0·9990	0·9780	0·7749	0·1730	−0·0806	−0·0991
0	0·9990	0·9782	0·7759	0·1743	−0·0804	−0·0991

Table II. Exact solution for v of two-dimensional Burgers' equation

y/y_{max}	$x=-1{\cdot}0$	$-0{\cdot}6$	$-0{\cdot}2$	$0{\cdot}2$	$0{\cdot}6$	$1{\cdot}0$
1·0	0·5774	0·5677	0·4611	0·1522	0·0206	0·0045
0·8	0·4452	0·4381	0·3593	0·1220	0·0167	0·0037
0·6	0·3249	0·3199	0·2642	0·0917	0·0127	0·0028
0·4	0·2126	0·2094	0·1738	0·0612	0·0085	0·0019
0·2	0·1051	0·1036	0·0862	0·0306	0·0043	0·0009
0	0·0	0·0	0·0	0·0	0·0	0·0

shown in Tables I and II. The following parameter values were chosen

$$k=0{\cdot}5, \quad R_e=5, \quad x_0=1, \quad a_0=a_1=0{\cdot}001\,k\exp((1+x_0)k)$$
$$y_{max}=\pi/(6k) \qquad a_2=a_3=0. \tag{15}$$

The solution shown in Table I is qualitatively similar to the one-dimensional 'shock' solution of Burgers' equation.

To illustrate the usefulness of equations (11) and (12), we have obtained steady-state solutions of equations (1) and (2) with a centred second-order finite difference formulation and a finite element formulation based on linear rectangular elements. The exact solutions, (11) and (12), supply both the boundary conditions for u and v and the final solution to test the relative accuracy of the finite difference and finite element methods. The steady-state solutions were obtained by the implicit integration of a time-split pseudotransient interpretation[6] of equations (1) and (2).

For various mesh sizes the rms errors in the finite difference and finite element solutions for u and v are shown in Table III.

Both methods converge like $O(\Delta^2 x, \Delta^2 y)$ approximately. The finite element formulation generates more accurate solutions for u particularly with a refined grid. The solution accuracy for v is comparable for the two formulations. The execution times per time step are appropriate to a CYBER–172. Both formulations produce execution times that are approximately linear in the number of grid points, as expected. However, the conventional finite element formulation is considerably less economical. This lack of economy can be traced to the relatively inefficient treatment of the convective terms, uu_x, vu_y, etc.

Table III. Comparison of finite difference (F.D.) and finite element (F.E.) solutions of equations (1) and (2)

Number of grids points	Δx $\Delta y/y_{max}$	u velocity, rms error F.D.	u velocity, rms error F.E.	v velocity, rms error F.D.	v velocity, rms error F.E.	Execution time per time step (secs) F.D.	Execution time per time step (secs) F.E.
6×6	0·4 0·2	$0{\cdot}130\times10^{-2}$	$0{\cdot}742\times10^{-3}$	$0{\cdot}323\times10^{-3}$	$0{\cdot}443\times10^{-3}$	0·0075	0·031
11×11	0·2 0·1	$0{\cdot}406\times10^{-3}$	$0{\cdot}169\times10^{-3}$	$0{\cdot}960\times10^{-4}$	$0{\cdot}980\times10^{-4}$	0·031	0·148
21×21	0·1 0·05	$0{\cdot}108\times10^{-3}$	$0{\cdot}319\times10^{-4}$	$0{\cdot}266\times10^{-4}$	$0{\cdot}247\times10^{-4}$	0·131	0·662
41×41	0·05 0·025	$0{\cdot}253\times10^{-4}$	$0{\cdot}274\times10^{-5}$	$0{\cdot}598\times10^{-5}$	$0{\cdot}578\times10^{-5}$	0·519	2·825

In this paper an economical procedure for generating exact solutions of the two-dimensional Burgers' equations has been described. It is expected that this will facilitate the use of the two-dimensional Burger's equations to test computational algorithms for solving the incompressible Navier–Stokes equations.

REFERENCES

1. C. A. J. Fletcher, 'Burgers' equation: a model for all reasons', in *Numerical Solution of Partial Differential Equations*, J. Noye (ed.), North-Holland, Amsterdam, 1982, pp. 139–225.
2. P. C. Jain and D. N. Holla, 'Numerical solution of coupled Burgers' equations', *Int. J. Nonlinear Mech.*, **13,** 213–222 (1978).
3. P. Arminjon and C. Beauchamp, 'Numerical solution of Burgers' equation in two space dimensions, *Int. J. Num. Meth. Eng.*, **12,** 415–428 (1978).
4. J. D. Cole, 'On a quasi-linear parabolic equation occurring in aerodynamics', *Quart. Appl. Math.*, **9,** 225–236 (1951).
5. W. F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Academic Press, New York, 1965, p. 23.
6. C. A. J. Fletcher, *Computational Galerkin Methods*, Springer-Verlag, Heidelberg, 1983.